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Remarks Concerning Completeness of Translates in Function Spaces

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This note explains how to translate the author's old result on cyclic vectors of the multiple shift operator into the language of completeness theorems for integer translates. This translation, together with those results, turns out to be a source for many completeness theorems. In particular, there follows the existence of functions f whose positive integer translates $f(x-k)$, where $k \in \mathbb{Z}_+$ are complete in the spaces $C_0^l(\mathbb{R})$, $L^p(\mathbb{R})$, $W_p^l(\mathbb{R})$, $2 < p < \infty$, $l = 0, 1, ...,$ as well as in their weighted and/or vector-valued analogues.

1. INTRODUCTION: SPECTRAL MULTIPLICITIES AND INTEGER TRANSLATES

Consider a Banach space X of functions (or distributions) on $\mathbb{R}$ that is translation invariant; that is, $\tau_s f \in X$ whenever $f \in X$ and $s \in \mathbb{R}$. Here

$$\tau_s f(x) = f(x-s)$$

stands for the translation operator. The problem we will address is: Does X contain a function $f \in X$ whose *integer* translates span the whole space, that is,

$$X = \operatorname{span}_X(\tau_s f : s \in \mathbb{Z}),$$

where span means the X-closed linear hull? The case of the completeness of positive integer translates $\tau_s f$, $s \in \mathbb{Z}_+$, is also of interest. Of course, this is a particular case of the general problem of finding the spectral multiplicity of a given operator; see the Appendix.

Functions satisfying these completeness properties are called $\mathbb{Z}$-*cyclic*, respectively $\mathbb{Z}_+$-*cyclic*, *elements*. The set of all $\mathbb{Z}$- or $\mathbb{Z}_+$-cyclic elements of

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X is denoted by $\mathrm{Cyc}(\mathbb{Z}, X)$ and $\mathrm{Cyc}(\mathbb{Z}_+, X)$, respectively. A space X is called $\mathbb{Z}$-*cyclic*, respectively $\mathbb{Z}_+$-*cyclic*, if $\mathrm{Cyc}(\mathbb{Z}, X) \neq \emptyset$, respectively, $\mathrm{Cyc}(\mathbb{Z}_+, X) \neq \emptyset$. We use *cyclic* for the cases where both the $\mathbb{Z}_+$- and $\mathbb{Z}$-cyclicity properties can be applied.

Clearly, the setting becomes more general if we consider a larger class of spaces X, namely, those X that are $\mathbb{Z}$-invariant only, that is, invariant with respect to τ_s, $s \in \mathbb{Z}$.

It is obvious that $\mathrm{Cyc}(\mathbb{Z}, X) \supset \mathrm{Cyc}(\mathbb{Z}_+, X)$ for every $\mathbb{Z}$-invariant space X. Moreover, such a space X can be identified with the space $\mathbf{X}$ of vector-valued sequences $(f_n)_{n \in \mathbb{Z}}$ of traces

$$f_n(t) = f(n+t), \qquad t \in [0, 1] \; (n \in \mathbb{Z}, f \in X)$$

endowed with the norm $\|(f_n)_{n \in \mathbb{Z}}\|_{\mathbf{X}} = \|f\|_X$. Clearly, the space $\mathbf{X}$ is τ_n, $n \in \mathbb{Z}$, invariant in an obvious sense. The problem stated above is then a special case of the following problem of cyclic (respectively, $\mathbb{Z}$- or $\mathbb{Z}_+$-*cyclic*) elements $f \in \mathbf{X}$ for the shift operator on $\mathbf{X}$: find an element $f \in \mathbf{X}$ such that $\mathbf{X} = \mathrm{span}_{\mathbf{X}}(\tau_n f : n \in \mathbb{Z})$, respectively, such that $\mathbf{X} = \mathrm{span}_{\mathbf{X}}(\tau_n f : n \in \mathbb{Z}_+)$.

Our principal (and trivial) observation is the following: Since $\tau_n X \subset X$, $n \in \mathbb{Z}$, the trace spaces $X_n = \{f_n : f \in X\}$ are identical: $X_n = X_0$ for every n. We call this space the *fiber space* of X. Hence, $\mathbf{X} = l_X(\mathbb{Z}, X_0)$ is a space of X_0-valued sequences $(f_n)_{n \in \mathbb{Z}}$. For the case where $l_X = l$ is a *lattice* (an ideal space; see the Appendix for precise definitions), the problem of cyclic vectors for $l(\mathbb{Z}, X_0)$ is solved in [N1, N2]. For the reader's convenience we reproduce in the Appendix those parts of the results of [N1, N2] that are needed here. For instance, it is proved that the answer does not depend on X_0 whenever $\dim(X_0) > 1$. This is one of the key points of the phenomenon. In other words, the cyclicity does not depend on the local structure of the space in question but on its asymptotic behaviour only.

The following lemma is useful for the case where $\mathbf{X} = l_X(\mathbb{Z}, X_0)$ is a closed subspace of a space $\mathbf{Y} = l(\mathbb{Z}, X_0)$ with a lattice l instead of l_X.

LEMMA 1.1. *A τ_n-invariant subspace $\mathbf{X} \subset \mathbf{Y}$ admitting a τ_n-invariant complement is cyclic if the space $\mathbf{Y}$ is cyclic.*

Proof. Let $\mathbf{X}'$ be an invariant complement of $\mathbf{X}$, and $\mathscr{P}$ the projection on $\mathbf{X}$ corresponding to the direct decomposition $\mathbf{Y} = \mathbf{X} + \mathbf{X}'$. Since $\tau_n \mathscr{P} = \mathscr{P} \tau_n$, the projection $\mathscr{P} f$ of a cyclic element $f \in \mathbf{Y}$ is cyclic for $\mathbf{X}$. The lemma follows.

In concluding this introduction let us mention, following [N1, N2], the meaning in terms of operator theory of the completeness of integer translates, and in particular of the results of Sections 2 and 3. The spectral theory of unitary operators shows obvious examples of *noncyclic* spaces, that is, $X = L^2(\mathbb{R}) = l^2(\mathbb{Z}, L^2(0, 1))$ and, more generally, any $\mathbf{X} = l^2(\mathbb{Z}, X_0)$ with a Hilbert space fiber X_0, $\dim(X_0) > 1$. It is also obvious that, in general, a space X, or $\mathbf{X}$, densely and continously embedded into the above noncyclic space $X \subset L^2(\mathbb{R})$, or equivalently $\mathbf{X} \subset l^2(\mathbb{Z}, X_0)$, is noncyclic itself. The meaning of the results quoted in the Appendix, and hence of the results from the next two sections, is that for a large class of spaces the latter embedding is the only reason for a space to be noncyclic.

The paper is organized as follows. As mentioned above, the Appendix contains a summary of the spectral multiplicities of the shift operators [N1, N2], which are crucial for the completeness results of Sections 2 and 3.

In Section 2 it is shown why many results on completeness of integer translates on $\mathbb{R}$, including those appearing in [AO], are, in the author's opinion, straightforward consequences of [N1, N2].

In Section 3 some $\mathbb{Z}^n$- and $(v \cdot \mathbb{Z})$-cyclicity results in $\mathbb{R}^n$ are obtained in the same way.

2. INTEGER TRANSLATES ON THE REAL LINE

The following examples are straightforward consequences of previous comments and the results presented in the Appendix.

EXAMPLE 2.1 (Unweighted Spaces). Each of the following spaces $C_0^l(\mathbb{R})$, $L^p(\mathbb{R})$, $W_p^l(\mathbb{R}, 2 < p < \infty, l = 0, 1, ...,$ is cyclic.

The spaces $L^p(\mathbb{R})$, $W_p^l(\mathbb{R})$ for $1 \leqslant p \leqslant 2$, $l = 0, 1, 2, ...$, are noncyclic.

Here

$$C_0(\mathbb{R}) = \{ f \in C(\mathbb{R}) : \lim_{|x| \to \infty} f(x) = 0 \},$$

$$C_0^l(\mathbb{R}) = \{ f : f^{(j)} \in C_0(\mathbb{R}), 0 \leqslant j \leqslant l \},$$

$$W_p^l(\mathbb{R}) = \{ f : f^{(j)} \in L^p(\mathbb{R}), 0 \leqslant j \leqslant l \},$$

the latter being the Sobolev spaces for $l \geqslant 1$, and $W_p^0 = L^p$.

These claims are immediate corollaries of the results found in the Appendix and the following simple lemma on the piecewise structure of the spaces in question. More precisely, the claimed properties for $L^p(\mathbb{R})$ and $W_p^l(\mathbb{R})$ are partial cases of Points 2 (for $1 \leqslant p \leqslant 2$) and 4 or 6 (for $2 < p < \infty$) of the Appendix, and for $C_0^l(\mathbb{R})$ of Point 4, 5, or 6.

LEMMA 2.2. (1) *The fiber space of* $L^p(\mathbb{R})$ *is* $L^p(0,1)$, *and* $L^p(\mathbb{R}) = l^p(\mathbb{Z}, L^p(0,1))$.

(2) *The fiber spaces of* $C_0^l(\mathbb{R})$ *and* $W_p^l(\mathbb{R})$ *are* $C^l([0,1])$ *and* $W_p^l([0,1])$, *respectively, and they are* $\mathbb{Z}$*-invariantly complemented subspaces of* $\mathbf{Y} = c_0(\mathbb{Z}, C^l([0,1]))$ *and* $\mathbf{Y} = l^p(\mathbb{Z}, W_p^l([0,1]))$, *respectively.*

Proof. The first claim is obvious. For the second one we simply write explicitly a τ_n-invariant projection from $\mathbf{Y} = c_0(\mathbb{Z}, C^l([0,1]))$ on $C_0^l(\mathbb{R})$. The latter is the subspace of $\mathbf{Y}$ consisting of sequences $(f_n) \in c_0(\mathbb{Z}, C^l([0,1])$ with the same Taylor polynomials in boundary points of adjacent intervals,

$$T_l(f_n, 1) = T_l(f_{n+1}, 0),$$

for all $n \in \mathbb{Z}$, where $T_l(f, \alpha)(t) = \sum_{k=0}^{l} (f^{(k)}(\alpha)/k!)\, t^k$. This description suggests the formula for the desired projection

$$\mathscr{P}_l(f_n) = (g_n),$$

where

$$g_n(t) = f_n(t) + [T_l(f_{n+1}, 0)(t-1) - T_l(f_n, 1)(t-1)]\, h(t), \qquad 0 \leqslant t \leqslant 1,$$

and h is an arbitrary smooth function smoothly taken values $h(0) = 0$, $h(1) = 1$. Indeed, $(g_n) \in c_0(\mathbb{Z}, C^l([0,1]))$ and $T_l(g_n, 1) = T_l(f_{n+1}, 0) = T_l(g_{n+1}, 0)$ for every n and every $(f_n) \in c_0(\mathbb{Z}, C^l([0,1]))$. On the other hand, $g_n = f_n$ for every $(f_n) \in C_0^l(\mathbb{R})$.

The Sobolev space $W_p^l(\mathbb{R}) \subset C^{l-1}(\mathbb{R})$ is a subspace of $\mathbf{Y} = l^p(\mathbb{Z}, W_p^l([0,1]))$ consisting of sequences (f_n) with $T_{l-1}(f_n, 1) = T_{l-1}(f_{n+1}, 0)$ for all $n \in \mathbb{Z}$. Hence, $\mathscr{P}_{l-1}$ is the required projection on the space $W_p^l(\mathbb{R})$. It is continuous since $|f^{(j)}(n)| \leqslant \int_n^{n+1} (|f^{(j+1)}| + |f^{(j)}|)\, dx$ for all $0 \leqslant j < l$ and all $n \in \mathbb{Z}$, which completes the proof.

EXAMPLE 2.3 (Weighted Spaces). Let

$$L^p(\mathbb{R}, w) = \{f : fw \in L^p(\mathbb{R})\}, \qquad 1 \leqslant p < \infty,$$

$$C_0^l(\mathbb{R}, w) = \{f : f^{(j)} \in C(\mathbb{R}), f^{(j)} w \in L^\infty(\mathbb{R}), f^{(j)} w = o(1) \text{ for } |x| \to \infty;\ 0 \leqslant j \leqslant l\},$$

$$W_p^l(\mathbb{R}, w) = \{f : f^{(j)} \in L^p(\mathbb{R}, w), 0 \leqslant j \leqslant l\}$$

where $w = w_0 w_1$ is a slowly varying function satisfying some natural hypotheses (see below).

(1) The space $C_0^l(\mathbb{R}, w)$ is cyclic if $w_0^2 = (w(n)^2)_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$, and is not if $w_0^{-2} \in l^1(\mathbb{Z})$.

(2) The spaces $L^p(\mathbb{R}, w)$ *and* $W_p^l(\mathbb{R}, w)$,

for $2 < p < \infty$, are cyclic if $w_0^2 \in l^\infty(\mathbb{Z})$ and are not if $w_0^{-2} \in l^{p/(p-2)}(\mathbb{Z})$;

for $p = 2$, are cyclic if $w_0 \in c_0(\mathbb{Z})$ and are not if $\inf_{n \in \mathbb{Z}} w_0(n) > 0$;

for $1 \leqslant p < 2$, are cyclic if $w_0^2 \in l^{p/(2-p)}(\mathbb{Z})$ and are not if $\inf_{n \in \mathbb{Z}} w_0(n) > 0$.

In these statements, cyclic means both $\mathbb{Z}$- or/and $\mathbb{Z}_+$-cyclic if the following condition is fulfilled:

$$\sum (n^2+1)^{-1} \, |\log \|\tau_n\| \, | < \infty, \quad (NQA)$$

and only $\mathbb{Z}$-cyclic otherwise.

Slowly varying weights are, by definition, positive functions of the form $w = w_0 w_1$, where w_1 is 1-periodic and w_0 is such that

$$\sup \left\{ \frac{w_0(x)}{w_0(y)} : |x-y| \leqslant 1 \right\} < \infty.$$

It is clear that $\sup_{x \in \mathbb{R}} (w(x+n)/w(x)) < \infty$ for a slowly varying weight w and for any $n \in \mathbb{Z}$, which means that the spaces of Example 2.3 are $\mathbb{Z}$-invariant.

In the case of C^l spaces, it is natural to suppose, and we do, that

$$\frac{1}{w_1} \in L^\infty(0, 1),$$

and all functions and their derivatives vanish at points where w_1, the local component of the weight, is unbounded.

In the case of W_p^l spaces, we suppose that

$$\frac{1}{w_1} \in L^{p'}(0, 1), \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

which implies $W_p^l(\mathbb{R}, w) \subset C^{l-1}(\mathbb{R})$.

Now, it is clear that $L^p(\mathbb{R}, w) = l((\mathbb{Z}, E)$, where $l = l^p(\mathbb{Z}, w_0(n))$ and $E = L^p((0, 1); w_1(x)\, dx)$. As in the previous example, the spaces $C_0^l(\mathbb{R}, w)$ and $W_p^l(\mathbb{R}, w)$ are the $\mathbb{Z}$-invariantly complemented subspaces of $l(\mathbb{Z}, E)$ with

$l = c_0(\mathbb{Z}, w_0(n))$, $E = C^l([0, 1], w_1)$ and $l = l^p(\mathbb{Z}, w_0(n))$, $E = W^l_p([0, 1], w_1)$, respectively. Hence the results claimed are special cases of Points 2–4 of the Appendix.

EXAMPLE 2.4 (Splines of Degree d). Given a function space X on $\mathbb{R}$, *let* XS^d be the space of X-splines of degree not exceeding $d \in \mathbb{Z}_+$, that is,

$$XS^d = \{ f \in X : f \mid [n, n+1] \text{ is a polynomial of degree } \leqslant d, \ \forall n \in \mathbb{Z} \}.$$

Let $d > 0$ and $X = L^p(\mathbb{R})$, $L^p(\mathbb{R}, w)$, or $d > 1$ and $X = C_0(\mathbb{R})$, $C_0(\mathbb{R}, w)$, where the spaces X are defined in Examples 2.1 and 2.3. Then the space XS^d is cyclic if and only if the space X is cyclic. In particular, the space of L^p splines $L^p S^d$, $d > 0$, is cyclic if and only if $p > 2$.

These claims are immediate corollaries of Point 1 of the Appendix and Examples 2.1 and 2.3.

The following three remarks comment on the results of the preceding examples and extend them in several directions.

Remark 2.5 (Vector-Valued Spaces, Mixed Normed Spaces, and Composition Operators). It is clear that the same cyclicity results hold for vector-valued function spaces, unweighted $L^p(\mathbb{R}, F)$, $C^l_0(\mathbb{R}, F)$, and $W^l_p(\mathbb{R}, F)$, where F stands for a separable Banach space, and weighted $L^p(\mathbb{R}, w, F)$, $C^l_0(\mathbb{R}, w, F)$, and $W^l_p(\mathbb{R}, w, F)$.

Examples 2.1, 2.3, and 2.4 can also be extended to a more general setting combining rearrangement-invariant sequences spaces l (see Point 6 of the Appendix) with various local spaces E. Classical Orlicz, Lorentz, and Marcinkiewicz spaces l can be used in any combination with the usual fiber spaces like $C^l([0, 1])$, $L^p(0, 1)$, $W^l_p[0, 1]$, etc.; the criteria of $\mathbb{Z}$- and $\mathbb{Z}_+$-cyclicity depend on the space l only. For example, the spaces

$$X_{p,q}(\mathbb{R}) = l^p(\mathbb{Z}, L^q(0, 1)) = \left\{ f : \sum_{n \in \mathbb{Z}} \| f_n \|_q^p < \infty \right\}$$

and

$$Y^l_p(\mathbb{R}) = l^p(\mathbb{Z}, C^l[0, 1])$$

are cyclic if and only if $p > 2$. By the way, the latter example implies the cyclicity of the space $X = C^l_0(\mathbb{R}) \cap L^p(\mathbb{R}), p > 2$, endowed with the norm $\| f \| = \| f \|_{C^l_0} + \| f \|_{Lp}$. Indeed, X densely contains the space $l^p(\mathbb{Z}, C^l[0, 1]) \cap C^l(\mathbb{R})$, that is, a complemented subspace of a cyclic space $Y^l_p(\mathbb{R})$.

We refer the reader to [N2] for more function spaces to be included in this approach. For instance, given an increasing function φ, $\lim_{x\to\infty}\varphi(x)=\infty$, the space $L^2_\varphi(\mathbb{R})=\{f : \int_{-x}^{x}|f(t)|^2\,dt=o(\varphi(x))\}$ is cyclic (see Point 6 of the Appendix), which contrasts with the non-cyclicity of $L^2(\mathbb{R})$.

Another application of the results of [N1, N2] is given by A. Kitover [K]. Namely, cyclicity is proved for the composition operators $C_\omega f=f\circ\omega$ and "skew" composition operators $f\mapsto M\cdot(f\circ\omega)$ for a class of homeomorphisms $\omega: K\to K$ of a metric compact K and unimodular functions $M\in C(K)$. These operators are considered on the subspace of $C(K)$; $C_{0,\omega}(K)=\{f\in C(K) : f \mid L(\omega)=0\}$, where (ω) stands for fixed points of ω. Typical examples are $\omega(t)=t^2$, $t\in K=[0,1]$ (with $L(\omega)=\{0\}\cup\{1\}$), and $\omega(t)=t+1$, $t\in K=\mathbb{R}\cup\{\infty\}$ (with $L(\omega)=\{\infty\}$), which again gives the $\mathbb{Z}$-cyclicity of $C_0(\mathbb{R})$. The proofs make use of the cyclicity of $c_0(\mathbb{Z},E)$ from [N1, N2], but in a different (more complicated) way than is done above.

Remark 2.6 ($\mathbb{Z}_+$-Cyclicity and the (NQA) Condition). Following the results of the Appendix and the notation in Section 1, the stronger property of $\mathbb{Z}_+$-cyclicity is guaranteed for the unweighted spaces of Examples 2.1 and 2.4, since the non-quasi–analiticity (NQA) condition is automatic. For the weighted spaces of Examples 2.3 and 2.4, the (NQA) condition is fulfilled if, for example, the asymptotic component of the weight (i.e., w_0) is even, supermultiplicative ($w_0(n+k)\geqslant C\cdot w_0(n)\,w_0(k)$), and $\sum_{n\geqslant 1} n^{-2}\log(1/w_0(n))<\infty$. Indeed, in this case the norms $\|\tau_n\|$ are comparable with $1/w_0(n)$.

Remark 2.7 ($\mathbb{Z}$-Cyclic Vectors, Multiple Shifts, and Simultaneous Approximations). As is shown above, $\mathbb{Z}$- and $\mathbb{Z}_+$-cyclicity results on the real line are often simple special cases of the spectral multiplicity theory for multiple shift operators constructed in [N1, N2]. The needed adaptation is usually reduced to an elementary change of notation, and one can say that, up to such a change of notation, and with the correct interpretation, these results are contained in [N1, N2]. The very problem of cyclic vectors for multiple shifts was first raised in [N1].

Let us point out two strong points of the vector-valued shift language proposed in [N1, N2] and used above. The first one has already been mentioned: the representation of translations τ_n acting on a function space X as the shift operators on the corresponding sequence space $\mathbf{X}=l_X(\mathbb{Z},X_0)$ allows one to separate the asymptotic properties of the space X (expressed in terms of the space l_X) for their crucial role for the problem, in contrast with the unessential local structure (i.e. with the space X_0). The second point is largely discussed in [N1, N2] and consists of a direct and straightforward coupling of the problem of cyclic vectors with the problem of simultaneous approximations; we refer the reader to the mentioned papers for details.

Moreover, one can interpret results on simultaneous approximations as spectral simplicity theorems for the shift operators; see for instance [Kh, N1, N2, N5] and further references therein. The simultaneous approximation meaning of the completeness of the integer translates explains the main reason for the mentioned completeness to hold true. Namely, such a reason consist in an incomparability of the $\mathscr{F}l$-convergence and the weakest local convergence, i.e., the convergence in measure; see [N2] and Point 2 of the Appendix. In particular, in the author's opinion, this is the key point of the existence results of the recent paper [AO], where some special cases of the above examples are presented (using a different approach), often without clear statements as to their relationship to [N1, N2].

3. REMARKS ON INTEGER TRANSLATES IN $\mathbb{R}^n$

The problem of integer translates completeness can also be considered in $\mathbb{R}^n$. It is possible to think about the completeness of the standard integer translates $\{\tau_s f : s \in \mathbb{Z}^n\}$ in a function space X, or translates with respect to another grid $\{\tau_s f : s \in G\}$ where G stands for another subgroup or subsemigroup of the translation group of $\mathbb{R}^n$, $G \subset \mathbb{R}^n$. The corresponding function space X is assumed to be invariant with respect to the translations τ_s, $s \in G$. A function f verifying the completeness property $\operatorname{span}_X(\tau_s f : s \in G) = X$ is said to be G-*cyclic*. The smaller G is, the stronger the corresponding G-cyclicity property. The strongest is for singly-generated discrete subgroups $\{\tau_{kv} : k \in \mathbb{Z}\} = v\mathbb{Z}$ and subsemigroups $\{\tau_{kv} : k \in \mathbb{Z}_+\} = v\mathbb{Z}_+$, where $v \in \mathbb{R}^n \backslash \{0\}$ is a fixed vector. The multiple shift approach of Section 1 and of [N1, N2] is well adapted to treating exactly this strongest cyclicity property. Below we list a few examples of such applications.

We restrict ourselves to the spaces $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$.

Example 3.1. Each of the spaces $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, $2 < p < \infty$, is $v\mathbb{Z}_+$-cyclic for every $v \in \mathbb{R}^n \backslash \{0\}$.

The spaces $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant 2$ are non-$v\mathbb{Z}$-cyclic, and even non-G-cyclic for every discrete subgroup $G \subset \mathbb{R}^n$.

Indeed, since these spaces are invariant with respect to the rotations and homotheties of $\mathbb{R}^n$, a function f is $v\mathbb{Z}$-cyclic if and only if it is $v'\mathbb{Z}$-cyclic for any pair $v, v' \in \mathbb{R}^n \backslash \{0\}$. The same equivalence is valid for subsemigroups $v\mathbb{Z}_+$. Hence we can set $v = \delta_n = (0, ..., 0, 1)$ and, as in Section 2, identify a function $f \in L^p(\mathbb{R}^n)$ with the sequence $(f_k)_{k \in \mathbb{Z}}$, where

$$f_k(x) = f(k\delta_n + x), \qquad x \in S_0, \quad k \in \mathbb{Z},$$

and $S_0 = \{x \in \mathbb{R}^n : 0 \leqslant x_n \leqslant 1\}$ is the unit strip in the direction δ_n. This procedure establishes a unitary bijection of the space $L^p(\mathbb{R}^n)$ onto the vector-valued sequence space $l^p(\mathbb{Z}, L^p(S_0))$, in such a way that τ_{nv}-translations in $L^p(\mathbb{R}^n)$ correspond to τ_n-translations in $l^p(\mathbb{Z}, L^p(S_0))$. Hence, the space $L^p(\mathbb{R}^n)$ is $v\mathbb{Z}_+$-cyclic for $2 < p < \infty$ (Point 4 of the Appendix), and is not $v\mathbb{Z}$-cyclic for $1 \leqslant p \leqslant 2$ (Point 2 of the Appendix).

For the spaces $C_0(\mathbb{R}^n)$ the same explanation holds as for $C_0(\mathbb{R})$ from Section 2, since it is a subspace of $c_0(\mathbb{Z}, C_0(S_0))$ having an invariant complement. Indeed, a $\mathbb{Z}$-invariant projection is given by the same formula as for the case of $\mathbb{R}$: $\mathscr{P}_0(f_n) = g_n)$, where

$$g_n(x) = g_n(x', x_n) = f_n(x', x_n) + [f_{n+1}(x', 0) - f_n(x', 1)]\, h(x_n),$$

$x = (x_1, ..., x_n) = (x', x_n) \in S_0$, and h is an arbitrary continuous function on $\mathbb{R}$ taking values $h(0) = 0$, $h(1) = 1$.

The $\mathbb{Z}^n$-non-cyclicity property of $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant 2$, is commented on in Remark 3.3 below.

Remark 3.2 (Using Vector-Valued Spaces). The same separation of variables $x = (x', x_n)$ and the Fubini theorem show that $L^p(\mathbb{R}^n) = L^p(\mathbb{R}, L^p(\mathbb{R}^{n-1}))$, and similarly $C_0(\mathbb{R}^n) = C_0(\mathbb{R}, C_0(\mathbb{R}^{n-1}))$. Hence, the result of Example 3.1 is a straightforward consequence of Remark 2.5 on the vector-valued version of the results of Section 2.

Remark 3.3 (About G-Cyclicity for a Discrete Subgroup G). To see that the spaces $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant 2$, are not G-cyclic for every discrete subgroup of $\mathbb{R}^n$, one can assume, without loss of generality, that the linear span of G coincides with $\mathbb{R}^n$, and hence there exists a linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(G) = \mathbb{Z}^n$. Since $L^p(\mathbb{R}^n) \circ A = L^p(\mathbb{R}^n)$, it suffices to show that $L^p(\mathbb{R}^n)$ is not $\mathbb{Z}^n$-cyclic for $1 \leqslant p \leqslant 2$. The latter property holds for exactly the same reason as the noncyclicity of $L^p(\mathbb{R}) = l^p(\mathbb{Z}, L^p(0, 1))$ does for $1 \leqslant p \leqslant 2$; see Point 2 of the Appendix. Namely, using the same idea as in Section 1, we easily identify

$$L^p(\mathbb{R}^n) = l^p(\mathbb{Z}^n, L^p(Q_0)),$$

where $Q_0 = \{x \in \mathbb{R}^n : 0 \leqslant x_i \leqslant 1, 1 \leqslant i \leqslant n\}$ stands for the unit cube of $\mathbb{R}^n$.

Now we proceed in the same way as we proved in [N2] the claims of Point 2 of the Appendix. Namely, starting with $p = 2$ and an arbitrary Hilbert fiber space E instead of $L^2(Q_0)$, and using the Plancherel theorem $\mathscr{F} l^2(\mathbb{Z}^n, E) = L^2(\mathbb{T}^n, E)$, we argue as follows: if $x \in l^2(\mathbb{Z}^n, E)$, then $\mathscr{F}(\tau_k x) = e^{ik \cdot t}(\mathscr{F} x)(e^{it})$ for every $k \in \mathbb{Z}^n$, and hence the E-valued functions $\mathscr{F}(\tau_k x)$ take values proportional to $\mathscr{F} x$. Therefore x cannot be a cyclic vector, if $\dim E > 1$.

For an arbitrary p, $1 \leqslant p \leqslant 2$, it suffices to prove the absence of $\mathbb{Z}^n$-cyclic vectors for $l^p(\mathbb{Z}^n, E)$ with a finite-dimensional fiber E, $\dim E > 1$ (having an arbitrary fiber space F and its finite-dimensional subspace E we apply the coordinate-wise projection from $l^p(\mathbb{Z}^n, F)$ on $l^p(\mathbb{Z}^n, E)$). Taking $p = 2$ and $E = \mathbb{C}^2$ and using the fact that $l^p(\mathbb{Z}^n, E)$ is dense in $l^2(\mathbb{Z}^n, E)$, we conclude that $l^p(\mathbb{Z}^n, F)$ has no $\mathbb{Z}^n$-cyclic vectors for $1 \leqslant p \leqslant 2$. This completes the verification of Example 3.1.

APPENDIX: MULTIPLE SHIFT WITH THE SIMPLE SPECTRUM

The following spectral simplicity problem is raised in [N1]: For which Banach space operators $T: X \to X$ do the direct products $T_n = T \times \cdots \times T$ have a simple spectrum; that is, for which T does there exist a T_n-cyclic vector $x = (x_1, ..., x_n) \in X^n$ such that

$$\operatorname{span}(T_n^k x : k \in \mathbb{Z}_+) = X^n?$$

This is equivalent to asking if for every $y = (y_1, ..., y_n) \in X^n$ and every $\varepsilon > 0$ there exists a polynomial p such that $\| p(T) x_i - y_i \| < \varepsilon$ simultaneously for all $1 \leqslant i \leqslant n$. For an invertible operator T the above vectors x are often called 1-*cyclic* to distinguish them from 2-*cyclic* vectors defined by $\operatorname{span}(T_n^k x : k \in \mathbb{Z}) = X^n$.

Below, we outline a theory constructed in [N1, N2] to answer this question for the shift operator $T = \tau_1$ on a sequence space $X = l$ on $\mathbb{Z}$. Clearly, this is a partial case of the more general problem of finding the spectral multiplicity (multicyclicity) of a given operator,

$$\mu(T) = \min\{\operatorname{card}(C) : C \subset X,\ \operatorname{span}(T^k C : k \geqslant 0) = X\},$$

and in particular of finding $\mu(T)$ for a direct product $T = T_1 \times \cdots \times T_n$. This latter multiplicity, limited by the natural bounds

$$\max_{1 \leqslant k \leqslant n} \mu(T_k) \leqslant \mu(T_1 \times \cdots \times T_n) \leqslant \sum_{1 \leqslant k \leqslant n} \mu(T_k),$$

is considered, in particular, in [N3, S, and N4], where the equality cases of these inequalities are studied and further references are given. In this notation, the question raised in [N1, N2] is equivalent to the following: When does $\mu(T \times \cdots \times T) = 1$?

In fact, instead of direct products $l \times \cdots \times l$ invariant with respect to the multiple shift $(\tau_1)_n = \tau_1 \times \cdots \times \tau_1$, we considered in [N1, N2] a more general class of $\mathbb{Z}$-invariant spaces $\mathbf{X} = l(\mathbb{Z}, E)$, where E is an arbitrary separable Banach space, $l = l(\mathbb{Z})$ is a $\mathbb{Z}$-invariant *ideal space* of complex sequences on $\mathbb{Z}$, and

$$\|(e_n)\|_{\mathbf{X}} = \|(\|e_n\|_E)_{n \in \mathbb{Z}}\|_l.$$

An *ideal space* is a lattice on $\mathbb{Z}$, that is,

$$|y_n| \leqslant |x_n|, \qquad n \in \mathbb{Z} \text{ and } x \in l, \text{ implies } y \in l, \ \|y\|_l \leqslant \|x\|_l.$$

The case of $(\tau_1)_n = \tau_1 \times \cdots \times \tau_1$ on $l \times \cdots \times l$ corresponds to the shift operator τ_1 on $l(\mathbb{Z}, \mathbb{C}^n)$. We always suppose that finitely supported sequences are dense in $l(\mathbb{Z})$.

The following is stated in [N1] and proved in [N2].

(1) *Everything depends on l.* Namely, if $l(\mathbb{Z}, \mathbb{C}^2)$ has a 1- or 2-cyclic vector, all of the $\mathbf{X} = l(\mathbb{Z}, E)$ do as well [N2, Theorem 6, p. 253 of the English translation]. The converse is obvious for every E, $\dim E \geqslant 2$, because $\mathbf{X}' = l(\mathbb{Z}, E')$ is complemented in $\mathbf{X} = l(\mathbb{Z}, E)$ for any $E' \subset E$, $\dim E' = 2$.

(2) *Embedding into* $\Sigma(\mathbb{T})$. For the space $l(\mathbb{Z}, \mathbb{C}^2)$, everything depends on the interrelations between the Fourier transform $\mathscr{F} l = \{\mathscr{F} x : x \in l\}$ and the metric space $\Sigma(\mathbb{T})$ of all measurable functions on the unit circle $\mathbb{T}$; here $\mathscr{F} x = \sum_{n \in \mathbb{Z}} x_n e^{int}$ for a finitely supported sequence x. In particular, *there are no 2-cyclic vectors in any* $l(\mathbb{Z}, E)$, $\dim E > 1$, *if* $\mathscr{F} l \subset \Sigma(\mathbb{T})$ *continously* [N2, Lemma 2, p. 242 of the English translation].

Examples. There exists no cyclic vector in $l(\mathbb{Z}, E)$, $\dim E > 1$, for τ_n-invariant spaces

$$l = l^p(\mathbb{Z}, w_n) = \{x \colon (x_n w_n)_{n \in \mathbb{Z}} \in l^p(\mathbb{Z})\}$$

embedded into $l^2(\mathbb{Z})$ (the latter is equivalent to $\inf_n w_n > 0$ for $1 \leqslant p \leqslant 2$, and to $\sum_n w_n^{-2p/(p-2)} < \infty$ for $2 < p \leqslant \infty$).

Note that for $p = \infty$ we always mean in our cyclicity assertions the separable subspace $c_0(\mathbb{Z}, w_n) = \{x : x_n w_n = o(1) \text{ for } |n| \to \infty\}$ instead of the formally written $l^\infty(\mathbb{Z}, w_n)$.

Three different approaches are proposed in [N1, N2] to exploit the mentioned interrelations between $\mathscr{F} l$ and $\Sigma(\mathbb{T})$; let us give an example of each of them.

For the sake of simplicity, we suppose (in this outline, but not everywhere in [N1] or [N2]) the following condition (A):

$$\mathscr{F} l \supset L^\infty(\mathbb{T}). \tag{A}$$

Note that for weighted spaces $l = l^p(\mathbb{Z}, w_n)$ this condition is equivalent to $w \in l^\infty(\mathbb{Z})$ for $p \geqslant 2$, and to $\sum_n w_n^{2p/(2-p)} < \infty$ for $p < 2$.

(3) *Approach* 1: *Weighted spaces l with a weight vanishing at infinity.* Let $\lim_{|n| \to \infty} \|\delta_n\|_l = 0$, where δ_n stands for the standard 0–1 basis of an ideal space l. *Then* $l(\mathbb{Z}, \mathbb{C}^2)$ (*and hence any* $l(\mathbb{Z}, E)$) *contains a* 2-*cyclic vector*. Moreover, under the non-quasi-analyticity condition,

$$\sum_{n \in \mathbb{Z}} (n^2+1)^{-1} \, |\log \|\tau_n\| \, | < \infty, \tag{NQA}$$

$l(\mathbb{Z}, \mathbb{C}^2)$ *contains a* 1-*cyclic vector as well* [N2, Theorem 3, p. 249 of the English translation].

Examples. All τ_n-invariant spaces $l = l^p(\mathbb{Z}, w_n)$ for $2 \leqslant p \leqslant \infty$ and $\lim_{|n| \to \infty} w_n = 0$, or $1 \leqslant p < 2$ and $\sum_n w_n^{2p/(2-p)} < \infty$ (see (A) above).

(4) *Approach* 2: *Spaces containing singular sequences.* Suppose l contains a nontrivial singular sequence x, that is, a sequence x with Fourier transform $\mathcal{F}x$ (which is a hyperdistribution on $\mathbb{T}$) supported on a subset of $\mathbb{T}$ having zero Lebesgue measure. Let the following technical condition hold: $l \subset l^\infty(\mathbb{Z}, 1/w_n)$ with an increasing logarithmically concave weight w satisfying the (NQA) hypothesis. *Then* $l(\mathbb{Z}, \mathbb{C}^2)$, *and hence all* $l(\mathbb{Z}, E)$, *contains a* 2-*cyclic vector* [N2, Theorem 4, p. 250 of the English translation].

Examples (Not Considered Before). (i) $l = l^p(\mathbb{Z})$, $2 < p < \infty$; $l = c_0(\mathbb{Z})$ or $l = l^\infty(\mathbb{Z})$ with weak* topology;

(ii) the corresponding weighted spaces $l = l^p(\mathbb{Z}, w_n)$ with $2 < p \leqslant \infty$ and $w \in l^\infty(\mathbb{Z})$;

(iii) the space $l^2_\varphi(\mathbb{Z}) = \{(c_k) : \sum_{k=-n}^{n} |c_k|^2 = o(\varphi(n)) \text{ for } n \to \infty\}$, where φ is a function tending to infinity.

(5) *Approach* 3: *Predual spaces of regular Banach algebras.* Let l^* be a regular convolution Banach algebra (from (A), with $\mathbb{T}$ being the maximal ideal space). *Then* $l(\mathbb{Z}, \mathbb{C}^2)$, *and hence all* $l(\mathbb{Z}, E)$, *contains* 2-*cyclic elements* [N2, Theorem 5, p. 251 of the English translation].

Examples. Again $l = c_0(\mathbb{Z})$, or $l = l^{p'}(\mathbb{Z}, w_n^{-1})$ with weights satisfying the $l^p(\mathbb{Z}, w_n)$ algebra condition (also obtained in [N2]).

(6) *More applications to classical function spaces.* Let l be a rearrangement-invariant sequence space on $\mathbb{Z}$, that is, $\|x \circ \omega\|_l = \|x\|_l$ for every $x \in l$ and every bijection $\omega : \mathbb{Z} \to \mathbb{Z}$ (in particular, l^p again, and Orlicz, Lorentz, and Marcinkiewicz spaces; see [KPS, LZ]). Recall that such a

space is completely defined by its *fundamental function* $\varphi_l(n) = \|\chi_A\|_l$, where $A \subset \mathbb{Z}$, card $A = n$. Without loss of generality, we can suppose thet φ_l is a concave function. The following corollaries of the above results contain most of the useful unweighted function spaces [N2, Corollaries 1 and 2, p. 257 of the English translation]:

(a) If $\sum_{n \geqslant 1} (\varphi_l(n+1) - \varphi_l(n))^2 < \infty$ then there exist 1- and 2-cyclic vectors in $l(\mathbb{Z}, E)$.

(b) If $\sum_{n \geqslant 1} (\psi_l(n+1) - \psi_l(n))^2 < \infty$, where $\psi_l(n) = n/\varphi_l(n)$ is the dual fundamental function, then there are neither 1- nor 2-cyclic vectors in $l(\mathbb{Z}, \mathbb{C}^2)$.

(c) If $\lim_{t \to 0} M(t)/t^2 = 0$ then the Orlicz space $l_M(\mathbb{Z}, E)$ contains 1- and 2-cyclic vectors.

(d) If $\underline{\lim}_{t \to 0} M(t)/t^2 > 0$ then there are neither 1- nor 2-cyclic vectors in $l_M(\mathbb{Z}, \mathbb{C}^2)$.

REFERENCES

[AO] A. Atzmon and A. Olevskii, Completeness of integer translates in function spaces on $\mathbb{R}$, *J. Approx. Theory* **87** (1996), 291–327.

[KH] S. V. Khrushchev (Hruscev), The problem of simultaneous approximations and removing of singularities of Cauchy integrals, *Trudy Mat. Inst. Stekl.* **130** (1978), 124–195; *Proc. Steklov Inst. Math.*, (1979) [Engl. transl.].

[K] A. Kitover, "Spectral Analysis of Isometric Operators in Spaces C and L^p," thesis, University of Leningrad, 1973 [in Russian].

[KPS] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, "Interpolation of Linear Operators," Nauka, Moscow, 1978 [in Russian]/American Math. Society, Providence, RI, 1982 [English transl.].

[LZ] W. A. J. Luxembourg and A. C. Zaanen, "Riesz Spaces," Vol. I, North-Holland, Amsterdam, 1971.

[N1] N. Nikolski, The multiple shift with a simple spectrum, *Zap. Nauch. Sem. LOMI* **19** (1970), 227–236 [in Russian]; *Sem. Math. V. A. Steklov Math. Inst. Leningrad* **19** (1970), 132–136. [Engl. transl.].

[N2] N. Nikolski, Selected problems of weighted approximation and spectral analysis, *Trudy Math. Inst. Steklova* **120** (1974) [in Russian]; *Proc. Steklov Math. Inst.* **120** (1974) [Engl. transl.].

[N3] N. Nikolski, Outlines for the computation of the multiplicities of the spectra of orthogonal sums, *Zap. Nauch. Semin. LOMI* **126** (1983), 150–158 [in Russian]; *J. Soviet Math.* **27** (1984), 2521–2526 [Engl. transl.].

[N4] N. Nikolski, Multicyclicity phenomenon. I. An introduction and maxi-formulas, *in* "Toeplitz Operators and Spectral Function Theory," Operator Theory: Advances and Applications, Vol. 42, pp. 9–57, Birkhäuser, Basel, 1989.

[N5] N. Nikolski, Lectures on the shift operator, IV, *Zap. Nauch. Semin. LOMI* **65** (1976), 103–132 [in Russian]; *J. Soviet Math.* **16** (1981), 1118–1139 [Engl. transl.].

[S] B. M. Solomyak, Multiplicity, calculi, and multiplication operators, *Sibir. Mat. Zh.* **29** (1988), 167–175 [in Russian]; *Siberian Math. J.* **29** (1988), 293–300 [Engl. transl.].